THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010H University Mathematics 2014-2015
Suggested Solution to Test 2

1. (a)

$$\begin{aligned} & \lim_{x\to 1/2} \frac{\cos^2 \pi x}{e^{2x} - 2ex} \qquad (\frac{0}{0}) \\ = & \lim_{x\to 1/2} \frac{-2\pi \cos \pi x \sin \pi x}{2e^{2x} - 2e} \qquad (\frac{0}{0}) \\ = & \lim_{x\to 1/2} \frac{-2\pi^2 \cos^2 \pi x + 2\pi^2 \sin^2 \pi x}{4e^{2x}} \\ = & \frac{\pi^2}{2e} \end{aligned}$$

(b) Let $y = (\sin x)^{\tan x}$, so $\ln y = \tan x \ln(\sin x) = \dfrac{\ln(\sin x)}{\cot x}$. Then

$$\begin{aligned} \lim_{x\to 0} \ln y &= \lim_{x\to 0} \frac{\ln(\sin x)}{\cot x} \qquad (\frac{\infty}{\infty}) \\ &= \lim_{x\to 0} (\frac{\cos x}{\sin x})/(-\csc^2 x) \\ &= \lim_{x\to 0} -\sin x \cos x \\ &= 0 \end{aligned}$$

Therefore, $\lim_{x\to 0} (\sin x)^{\tan x} = e^0 = 1$

2. Let $y = \tan^{-1} x$, that means $\tan y = x$ and so $\cos y = \dfrac{1}{\sqrt{1+x^2}}$. Then

$$\begin{aligned} \tan y &= x \\ \sec^2 y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \cos^2 y \\ \frac{d}{dx} \tan^{-1} x &= \frac{1}{1+x^2} \end{aligned}$$

3. We have

$$\begin{aligned} f(x) &= \ln(1+x) \\ f'(x) &= \frac{1}{1+x} \\ f''(x) &= \frac{-1}{(1+x)^2} \\ f'''(x) &= \frac{2}{(1+x)^3} \end{aligned}$$

Therefore, $f(0) = 0$, $f'(0) = 1$, $f''(0) = -1$ and $f'''(0) = 2$ and

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = x - \frac{x^2}{2} + \frac{x^3}{3}$$

Then

$$\ln 1.01 = f(0.01) \approx P_3(0.01) = 0.009950333$$

4. (a) Let $f(t) = (1-k)t + k - t^{1-k}$, then $f'(t) = (1-k) - (1-k)t^{-k} = (1-k)(1-t^{-k})$. Note $0 < k < 1$, so $1-k > 0$.

$$\begin{aligned} f'(t) &> 0 \\ (1-k)(1-t^{-k}) &> 0 \\ 1 &> t^{-k} \\ t^k &> 1 \\ t &> 1 \end{aligned}$$

Similarly, $f'(t) < 0$ when $0 < t < 1$.

Therefore, $f(t) \geq f(1) = 0$ for all $t > 0$ and it follows that $(1-k)t + k \geq t^{1-k}$ for all $t > 0$.

(b) Since $r, s > 0$, $\frac{r}{s} > 0$. Using the result in (a),

$$\begin{aligned} (1-k)(\frac{r}{s}) + k &\geq (\frac{r}{s})^{1-k} \\ (1-k)r + ks &\geq r^{1-k}s^k \end{aligned}$$

5. (a) $f'(x) = (1-2x^2)e^{-x^2}$ and $f''(x) = 2x(2x^2-3)e^{-x^2} = 2x(\sqrt{2}x - \sqrt{3})(\sqrt{2}x + \sqrt{3})e^{-x^2}$.

(b) Note that $e^{-x^2} > 0$

(i) $f'(x) > 0$ when $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

(ii) $f'(x) < 0$ when $x < -\frac{1}{\sqrt{2}}$ or $x > \frac{1}{\sqrt{2}}$

(iii) $f''(x) > 0$ when $-\sqrt{\frac{3}{2}} < x < 0$ or $x > \sqrt{\frac{3}{2}}$

(iv) $f''(x) < 0$ when $x < -\sqrt{\frac{3}{2}}$ or $0 < x < \sqrt{\frac{3}{2}}$

(c) $f(x)$ has a local minimum point $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}e^{-1/2})$ and a local maximum point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}e^{-1/2})$.

(d) $f(x)$ has points of inflections $(-\sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}}e^{-3/2})$, $(0,0)$ and $(\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}e^{-3/2})$.

(e) Note $m = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} e^{-x^2} = 0$, and $c = \lim_{x\to\infty} f(x) - mx = \lim_{x\to\infty} xe^{-x^2} = 0$.

Therefore, $f(x)$ has a horizontal asymptote $y = 0$.

(f) The graph of $f(x)$.